Calculus II - Day 23

Prof. Chris Coscia, Fall 2024
Notes by Daniel Siegel

4 December 2024

Goals for Today

- Compute tangent lines to polar functions and determine when these lines are vertical or horizontal.
- Compute areas bounded by polar functions $r = f(\theta)$ and arc lengths of polar functions.

Finding the Slope of the Tangent Line

Given a polar function $r = f(\theta)$, how do we find the slope of the tangent line? We want $\frac{dy}{dx}$.

Recall

$$x = r\cos(\theta) = f(\theta)\cos(\theta), \quad y = r\sin(\theta) = f(\theta)\sin(\theta)$$

Both x and y are parametric equations. If $x = f(t)$ and $y = g(t)$, then:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

In Polar Coordinates

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{(f(\theta)\sin(\theta))'}{(f(\theta)\cos(\theta))'}$$

$$\boxed{\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}}$$

Example: Tangent Lines to a Circle

Given $r = 5$, we have:

$$f(\theta) = 5, \quad f'(\theta) = 0$$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

$$\frac{dy}{dx} = \frac{0 \cdot \sin(\theta) + 5 \cdot \cos(\theta)}{0 \cdot \cos(\theta) - 5 \cdot \sin(\theta)} = \frac{5\cos(\theta)}{-5\sin(\theta)} = -\cot(\theta)$$

Note: This result is the same for every circle centered at the origin.

Undefined Derivative

$-\cot(\theta)$ is not always defined on the interval $[0, 2\pi]$. Specifically:

$$\text{No derivative at } \theta = 0 \text{ or } \theta = \pi$$

At these points, the tangent lines are vertical.

Finding Points of Vertical and Horizontal Tangency

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

- $\frac{dy}{d\theta} = 0 \quad (\text{and } \frac{dx}{d\theta} \neq 0)$: Horizontal tangent.
- $\frac{dx}{d\theta} = 0 \quad (\text{and } \frac{dy}{d\theta} \neq 0)$: Vertical tangent.
- If $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} = 0$: The tangent could be vertical, horizontal, or neither.
  - In this case, use L'Hôpital's rule to determine the behavior of $\frac{dy}{dx}$.

The Cardioid $r = f(\theta) = 1 + \sin(\theta)$, $0 \leq \theta \leq 2\pi$

We find the vertical and horizontal tangents for the cardioid.

$$\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

$$\frac{dy}{dx} = \frac{\cos(\theta)\sin(\theta) + (1 + \sin(\theta))\cos(\theta)}{\cos^2(\theta) - (1 + \sin(\theta))\sin(\theta)}$$

Simplify numerator and denominator:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos(\theta)(1 + 2\sin(\theta))}{1 - \sin^2(\theta) - \sin(\theta) - \sin^2(\theta)} \\ &= \frac{\cos(\theta)(1 + 2\sin(\theta))}{1 - \sin(\theta) - 2\sin^2(\theta)} \\ &= \frac{\cos(\theta)(1 + 2\sin(\theta))}{(1 - 2\sin(\theta))(1 + \sin(\theta))}.\end{aligned}$$

Set the Numerator Equal to Zero: Horizontal Tangents

$$\cos(\theta)(1 + 2\sin(\theta)) = 0$$
$$\cos(\theta) = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$
$$1 + 2\sin(\theta) = 0 \quad \Rightarrow \quad \sin(\theta) = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

Set the Denominator Equal to Zero: Vertical Tangents

$$(1 - 2\sin(\theta))(1 + \sin(\theta)) = 0$$
$$1 - 2\sin(\theta) = 0 \quad \Rightarrow \quad \sin(\theta) = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$1 + \sin(\theta) = 0 \quad \Rightarrow \quad \sin(\theta) = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{2}.$$

Results

- **Horizontal tangents:** $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
- **Vertical tangents:** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

What Happens at $\theta = \frac{3\pi}{2}$?

Check: $\lim_{\theta \to \frac{3\pi}{2}} \frac{dy}{dx}$

To determine the behavior of the tangent line at $\theta = \frac{3\pi}{2}$, evaluate:

$$\lim_{\theta \to \frac{3\pi}{2}} \frac{\cos(\theta)(1 + 2\sin(\theta))}{(1 - 2\sin(\theta))(1 + \sin(\theta))}.$$

At $\theta = \frac{3\pi}{2}$, the expression is of the indeterminate form $\frac{0}{0}$. Apply L'Hôpital's Rule:
Differentiate the numerator:

$$\frac{d}{d\theta}\big[\cos(\theta)(1 + 2\sin(\theta))\big] = -\sin(\theta)(1 + 2\sin(\theta)) + \cos(\theta)(2\cos(\theta)).$$

Differentiate the denominator:

$$\frac{d}{d\theta}\big[(1 - 2\sin(\theta))(1 + \sin(\theta))\big] = (-2\cos(\theta))(1 + \sin(\theta)) + (1 - 2\sin(\theta))\cos(\theta).$$

Substitute these derivatives into the limit:

$$\lim_{\theta \to \frac{3\pi}{2}} \frac{-\sin(\theta)(1 + 2\sin(\theta)) + 2\cos^2(\theta)}{-2\cos(\theta)(1 + \sin(\theta)) + \cos(\theta)(1 - 2\sin(\theta))}.$$

At $\theta = \frac{3\pi}{2}$:

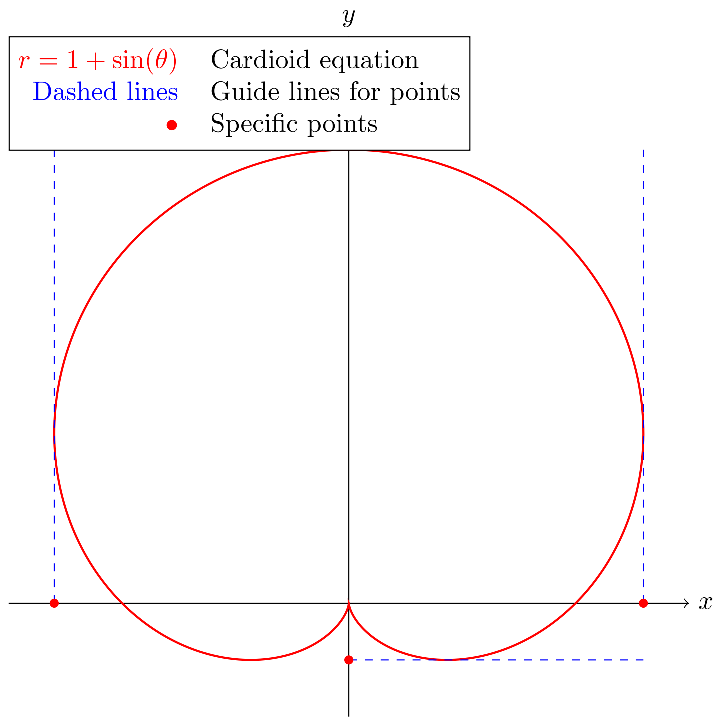
$$\text{Numerator:} \; -(-1)(1 + 2(-1)) + 2(0)^2 = 1 - 2 = -1.$$
$$\text{Denominator:} \; -2(0)(1 + (-1)) + (0)(1 - 2(-1)) = 0.$$

The limit becomes:

$$\frac{-1}{0}.$$

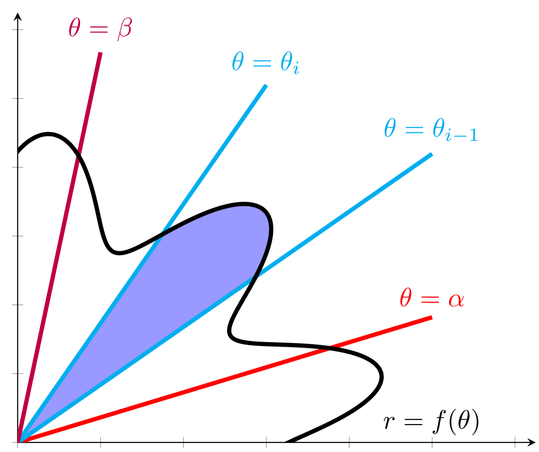
Conclusion

The limit diverges (DNE). Therefore, the tangent line at $\theta = \frac{3\pi}{2}$ is vertical.

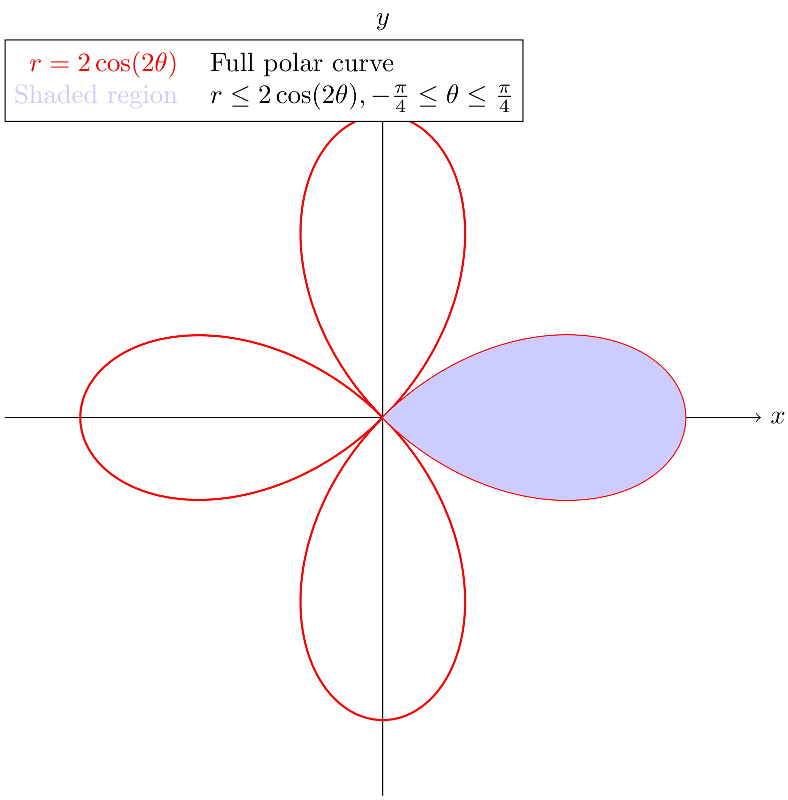


Area Bounded by a Polar Curve

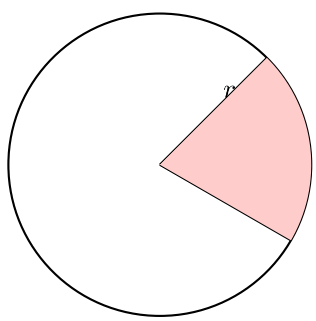
Estimate using "wedges": sector of a circle (instead of Riemann sums)



Another Example of Bounded Region



Area of a Sector of a Circle



Area of a sector of a circle:

$$\text{Area} = \text{Area of circle} \cdot \frac{\Delta\theta}{2\pi}$$

$$= \pi(r_x)^2 \cdot \frac{\Delta\theta}{2\pi}$$

$$= \frac{1}{2}(r_x)^2\Delta\theta$$

Area Bounded by a Polar Curve

For a polar curve $r = f(\theta)$ over $\theta \in [\alpha, \beta]$, the area is given by:

$$\text{Approximation: } \sum_{k=1}^{n} \frac{1}{2}\big(f(\theta_k)\big)^2\Delta\theta \quad \text{as } n \to \infty.$$

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta.}$$

Example: Find the Area Enclosed by One Petal of the 4-Petaled Rose

Given $r = 2\cos(2\theta)$, the area of one petal is:

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2}\big(2\cos(2\theta)\big)^2 \, d\theta.$$

$$= \int_{-\pi/4}^{\pi/4} 2\cos^2(2\theta)\, d\theta.$$

Using the identity $\cos^2(x) = \frac{1+\cos(2x)}{2}$:

$$A = \int_{-\pi/4}^{\pi/4} 2 \cdot \frac{1+\cos(4\theta)}{2}\, d\theta = \int_{-\pi/4}^{\pi/4} \big(1 + \cos(4\theta)\big)\, d\theta.$$

$$A = \int_{-\pi/4}^{\pi/4} 1\, d\theta + \int_{-\pi/4}^{\pi/4} \cos(4\theta)\, d\theta.$$

$$A = \theta + \frac{1}{4}\sin(4\theta)\Big|_{-\pi/4}^{\pi/4}.$$

Evaluate:

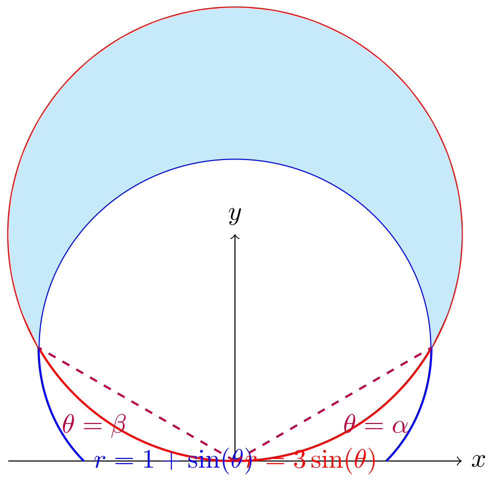
$$A = \left(\frac{\pi}{4} + \frac{1}{4}\cdot 0\right) - \left(-\frac{\pi}{4} + \frac{1}{4}\cdot 0\right) = \frac{\pi}{4} + \frac{\pi}{4}.$$

$$A = \boxed{\frac{\pi}{2}}.$$

Area Between Two Polar Curves

For two polar curves $r = f(\theta)$ and $r = g(\theta)$, the area is:

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\big(f(\theta)^2 - g(\theta)^2\big)\, d\theta.$$



$r = 3\sin(\theta)$	Circle equation
$r = 1 + \sin(\theta)$	Cardioid equation
Shaded region	Area above $r = 1 + \sin(\theta)$ and below $r = 3\sin(\theta)$
Dashed lines	$\theta = \alpha, \theta = \beta$

Example: Find the Area Inside the Circle $r = 3\sin(\theta)$ and Outside the Cardioid $r = 1 + \sin(\theta)$

Set the curves equal to find the limits of integration:

$$3\sin(\theta) = 1 + \sin(\theta) \quad \Rightarrow \quad \sin(\theta) = \frac{1}{2}.$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

The area is:

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left((3\sin(\theta))^2 - (1+\sin(\theta))^2\right) d\theta.$$

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(9\sin^2(\theta) - (1 + 2\sin(\theta) + \sin^2(\theta))\right) d\theta = \int_{\pi/6}^{5\pi/6} \left(\frac{3}{2} - 2\cos(2\theta) - \sin(\theta)\right) d\theta.$$

$$A = \left[\frac{3}{2}\theta - \sin(2\theta) + \cos(\theta)\right]_{\pi/6}^{5\pi/6}.$$

$$A = \frac{3}{2} \cdot \frac{4\pi}{6} - (0 - 0) + \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \pi.$$

Polar Arc Length

The length of the arc traced by $r = f(\theta)$ on the interval $\theta \in [\alpha, \beta]$ is

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + \left(f'(\theta)\right)^2}\, d\theta$$

Where does this come from? **Parametric arc length:**

$$L = \int_{a}^{b} \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\, dt$$

where $x = f(t)$ and $y = g(t)$.

In polar coordinates:

$$x = f(\theta)\cos(\theta), \quad y = f(\theta)\sin(\theta)$$

Example: Find the arc length of the spiral graph $f(\theta) = e^{\theta}$, $0 \leq \theta \leq t$:

$$L = \int_0^t \sqrt{f(\theta)^2 + \left(f'(\theta)\right)^2}\, d\theta = \int_0^t \sqrt{2e^{2\theta}}\, d\theta = \int_0^t \sqrt{2}e^{\theta}\, d\theta = \sqrt{2}e^{\theta}\Big|_0^t = \sqrt{2}e^t - \sqrt{2}.$$